Ring Theory 16

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Q> Tell whether $\mathbb{Z}[x, y]$ is a UFD, PID, ED or not?

Ans:- The ideal $\langle x, y \rangle$ is not principal $\Rightarrow$ not a PID

$\mathbb{Z}$ is a UFD $\Rightarrow$ $\mathbb{Z}[x]$ is a UFD $\Rightarrow$ $\mathbb{Z}[x][y]$ is a UFD

and $\mathbb{Z}[x][y] \cong \mathbb{Z}[x, y]$ $\Rightarrow$ it is UFD

It is not PID $\Rightarrow$ it is not ED

Polynomial Rings:-

Let R be a ring and we define a polynomial $P(x)$ over R in the form $P(x) = \sum_{i=0}^{n} a_i x^i$; $a_i \in R$ and $a_n \neq 0$, $d(P) = n$

$P(x) \in R[x]$

"x" is called an indeterminate.

Properties:-

- $R[x]$ is a ring under both operations
- If R is commutative $\Leftrightarrow$ $R[x]$ is commutative
- Unity of $R[x]$ is $1 \in R$
- If R is an integral domain $\Leftrightarrow$ $R[x]$ is also an integral domain
- If F is a field then $F[x]$ need not be field. But $F[x]$ is an integral domain

Factor Theorem for Ring:- If R is commutative ring $(x-a)$ is a factor of $P(x) \in R[x]$ iff a is a zero in and $a \in R$

Bifactor Theorem. – Let $a, b \in$ commutative ring R and $P(x) \in R[x]$.

Suppose $P(a) = P(b) = 0 \iff P(x) = (x-a)(x-b)Q$
$Q \in R[x]$

$(x^2-1) \bmod 8$

$a^2-1-(b^2-1) \equiv 0$

$a - b \equiv 0$

$a^2 - b^2 \equiv 0$

$(a+b)(a-b) \equiv 0$

a^2-1-b^2-1

$a^2-1 \equiv 0$

$P(b) = 0 \Rightarrow P(x) = (x-b)Q_b \quad Q_b \in R[x]$

$P(a) = (a-b)Q_b$ if $a - b = 0 \Rightarrow a = b$

$= 0$

- If $P(x)$ is a polynomial of degree n then $P(x)$ has at most n roots → This statement is not true over arbitrary rings

$\Rightarrow P(x) = (x-a)Q_a$ ∵ if a is a zero in $P(x)$ and $a \in R$

if any other root b is to be found, it should be in Q_a

$\Rightarrow Q_a = (x-b)Q_b \quad \Rightarrow P(x) = (x-a)(x-b)Q_b$

$P(x) = (x-b)Q_{b'}$

$P(a) = 0, \ P(b) = 0$

$P(b) = (b-a)Q_a$ $\quad$ a, b are distinct $\Rightarrow Q_a = 0$

↳ but we can't say that $(a-b)Q_a = 0$

$P(a) = (a-b)Q_{b'}$

⟶ This statement will hold for comm R

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- A non-zero polynomial $f(x) \in F[x]$ of degree n have at most n zeros.

Defn:- If F is a field and $f(x) \in F[x]$ is a non-constant polynomial. $f(x)$ is reducible over F iff it can be factored as a product of two non-constant polynomials in $F[x]$. Otherwise it is irreducible

Theorem:- Let $F[x]$ is a polynomial ring over F

If $f, g, P_1, P_2 \in F[x]$ and f is non-zero then
$fg = 0 \Rightarrow g = 0$ and $fP_1 = fP_2 \Rightarrow P_1 = P_2$

f has an inverse in $F[x]$ iff f is a non-zero constant.

Division Algorithm:- $F[x]$ is a polynomial ring and F is a field

Then $f = qd + r$ where $\deg(r) < \deg(d)$ or $r = 0$ and q, r are unique for f.